World Spinors: Group Representations and Field Equations

Djordje Šijački
Institute of Physics, P.O. Box 57, 11001 Belgrade, Serbia

Abstract

The questions of the existence, basic algebraic properties and relevant constraints that yield a viable physical interpretation of world spinors are discussed in details. Relations between spinorial wave equations that transform respectively w.r.t. the tangent flat-space (anholonomic) Affine symmetry group and the world generic-curved-space (holonomic) group of Diffeomorphisms are presented. A geometric construction based on an infinite-component generalization of the frame fields (e.g. tetrads) is outlined. The world spinor field equation in $3D$ is treated in more details.

1 Introduction.

The Dirac equation turned out to be one of the most successful equations of the XX century physics - it describes the basic matter constituents (both particles and fields), and very significantly, it paved a way to develop the concept of gauge theories thus completing the description of the basic interactions as well. It is a Poincaré invariant linear field equation which describes relativistic spin $\frac{1}{2}$ particles, with interactions naturally introduced by the minimal coupling prescription.

Here we go **beyond the Poincaré invariance** and study **affine invariant generalizations** of the Dirac equation. In other words, we consider a **generalization that will describe a spinorial field - world spinors - in a generic curved spacetime** (L_n, g), **characterized by arbitrary torsion and general-linear curvature**. Note that the spinorial fields in the non-affine generalizations of GR (which are based on higher-dimensional

orthogonal-type generalizations of the Lorentz group) are only allowed for special spacetime configurations and fail to extend to the generic case.

The **finite-dimensional world tensor fields** in R^n are characterized by the non-unitary irreducible representations of the general linear subgroup $GL(n,R)$ of the Diffeomorphism group $Diff(n,R)$. In the flat-space limit they split up into $SO(1,n-1)$ ($SL(2,C)/Z_2$ for $n=4$) irreducible pieces. The corresponding particle states are defined in the tangent flat-space only. They are characterized by the unitary irreducible representations of the (inhomogeneous) Poincaré group $P(n) = T_n \wedge SO(1,n-1)$, and they are defined by the relevant "little" group unitary representation labels.

In the **generalization to world spinors**, the double covering group, $\overline{SO}(1,n-1)$, of the $SO(1,n-1)$ one, that characterizes a Dirac-type fields in $D=n$ dimensions, is enlarged to the $\overline{SL}(n,R) \subset \overline{GL}(n,R)$ group,

$$\overline{SO}(1,n-1) \quad \mapsto \quad \overline{SL}(n,R) \subset \overline{GL}(n,R)$$

while $SA(n,R) = T_n \wedge \overline{SL}(n,R)$ is to replace the Poincaré group itself.

$$P(n) \quad \mapsto \quad \overline{SA}(n,R) = T_n \wedge \overline{SL}(n,R)$$

Affine "particles" are characterized by the unitary irreducible representations of the $\overline{SA}(n,R)$ group, that are actually nonlinear unitary representations over an appropriate "little" group. E.g. for $m \neq 0$:

$$T_{n-1} \otimes \overline{SL}(n-1,R) \supset T_{n-1} \otimes \overline{SO}(n)$$

A mutual **particle–field correspondence** is achieved by requiring **(i)** that fields have appropriate mass (Klein-Gordon-like equation condition, for $m \neq 0$), and **(ii)** that the subgroup of the field-defining homogeneous group, which is isomorphic to the homogeneous part of the "little" group, is represented unitarily. Furthermore, one has to project away all representations except the one that characterizes the particle states.

A **physically correct picture, in the affine case, is obtained by making use of the $\overline{SA}(n,R)$ group unitary (irreducible) representations for "affine" particles.** *The affine-particle states are characterized by the unitary (irreducible) representations of the $T_{n-1} \otimes \overline{SL}(n-1,R)$ "little" group. The intrinsic part of these representations is* **necessarily infinite-dimensional due to non-compactness of the $SL(n,R)$ group.** *The corresponding affine fields should be described by the non-unitary infinite-dimensional $\overline{SL}(n,R)$ representations, that are* **unitary** *when restricted to*

the homogeneous "little" subgroup $\overline{SL}(n-1,R)$. Therefore, **the first step towards world spinor fields is a construction of infinite-dimensional non-unitary $\overline{SL}(n,R)$ representations, that are unitary when restricted to the $\overline{SL}(n-1,R)$ group. These fields reduce to an infinite sum of (non-unitary) finite-dimensional $\overline{SO}(1,n-1)$ fields**.

2 Existence of the double-covering $\overline{GL}(n,R)$.

Let us state first some relevant mathematical results.
Theorem 1: Let $g_0 = k_0 + a_0 + n_0$ be an Iwasawa decomposition of a semisimple Lie algebra g_0 over R. Let G be any connected Lie group with Lie algebra g_0, and let K, A, N be the analytic subgroups of G with Lie algebras k_0,a_0 and n_0 respectively. The mapping $(k,a,n) \to kan$ $(k \in K, a \in A, n \in N)$ is an analytic diffeomorphism of the product manifold $K \times A \times N$ onto G, and *the groups* A *and* N *are simply connected.*

Any semisimple Lie group can be decomposed into the product of the *maximal compact subgroup* K, an *Abelian group* A and a *nilpotent group* N. **As a result of Theorem 1, only K is not guaranteed to be simply-connected**. There exists a universal covering group $\overline{K}$ of K, and thus also a universal covering of G: $\overline{G} \simeq \overline{K} \times A \times N$.

For the group of diffeomorphisms, let $Diff(n,R)$ be the group of all homeomorphisms f or R^n such that f and f^{-1} are of class C^1. Stewart proved the decomposition $Diff(n,R) = GL(n,R) \times H \times R_n$, where the subgroup H is contractible to a point. Thus, as $O(n)$ is the compact subgroup of $GL(n,R)$, one finds

Theorem 2: $O(n)$ is a deformation retract of $Diff(n,R)$.

As a result, there exists a universal covering of the Diffeomorphism group $\overline{Diff}(n,R) \simeq \overline{GL}(n,R) \times H \times R_n$.

Summing up, we note that both $SL(n,R)$ and on the other hand $GL(n,R)$ and $Diff(n,R)$ will all have double coverings, defined by $\overline{SO}(n)$ and $\overline{O}(n)$ respectively, the double-coverings of the $SO(n)$ and $O(n)$ maximal compact subgroups.

In the physically most interesting case $n = 4$, there is a homomorphism between $SO(3) \times SO(3)$ and $SO(4)$. Since $SO(3) \simeq SU(2)/Z_2$, where Z_2 is the two-element center $\{1,-1\}$, one has $SO(4) \simeq [SU(2) \times SU(2)]/Z_2^d$, where Z_2^d is the diagonal discrete group whose representations are given by $\{1, (-1)^{2j_1} = (-1)^{2j_2}\}$ with j_1 and j_2 being the Casimir labels of the two $SU(2)$ representations. The full $Z_2 \times Z_2$ group, given by the representations $\{1,(-1)^{2j_1}\} \otimes \{1,(-1)^{2j_2}\}$, is the center of $\overline{SO}(4) = SU(2) \times SU(2)$, which

is thus the quadruple-covering of $SO(3) \times SO(3)$ and a double-covering of $SO(4)$. $SO(3) \times SO(3)$, $SO(4)$ and $\overline{SO}(4) = SU(2) \times SU(2)$ are thus the maximal compact subgroups of $SO(3,3)$, $SL(4,R)$ and $\overline{SL}(4,R)$ respectively. One can sum up these results by the following exact sequences

$$\begin{array}{ccccccccc}
 & & & & 1 & & 1 & & \\
 & & & & \downarrow & & \downarrow & & \\
1 & \rightarrow & Z_2^d & \rightarrow & Z_2 \times Z_2 & \rightarrow & Z_2 & \rightarrow & 1 \\
 & & & & \downarrow & & \downarrow & & \\
1 & \rightarrow & Z_2^d & \rightarrow & \overline{SL}(4,R) & \rightarrow & SL(4,R) & \rightarrow & 1 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & SO(3,3) & & SO(3,3) & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 1 & & 1 & &
\end{array}$$

3 The $SL(4,R) \rightarrow SL(4,C)$ embedding.

A glance at the classical semisimple Lie group Dynkin diagrams tells us that we need to investigate two possibilities: either one can embed the $\overline{SL}(4,R)$ algebra $sl(4,R)$ in the Lie algebra of the appropriate noncompact version of the orthogonal algebra $so(6)$ of the $Spin(6)$ group, or in the $sl(4,C)$ algebra of the $SL(4,C)$ group. In the first case, the appropriate noncompact group is $Spin(3,3) \simeq \overline{SO}(3,3)$ which is *isomorphic* to the $SL(4,R)$ group itself. As for the second option, we consider first the problem of embedding the algebra $sl(4,R) \rightarrow sl(4,C)$.

The maximal compact subalgebra $so(4)$ of the $sl(4,R)$ algebra is embedded into the maximal compact subalgebra $su(4)$ of the $sl(4,C)$ algebra.

$$\begin{array}{ccc}
sl(4,R) & \rightarrow & sl(4,C) \\
\cup & & \cup \\
so(4) & \rightarrow & su(4)
\end{array}$$

There are two principally different ways to carry out the $so(4) \rightarrow su(4)$.

Natural $(\frac{1}{2},\frac{1}{2})$ embedding

In this embedding the $so(4)$ algebra is represented by the genuine 4×4 orthogonal matrices of the 4-vector $(\frac{1}{2},\frac{1}{2})$ representation (i.e. antisymmetric matrices multiplied by the imaginary unit). The $SL(4,C)$ generators split with respect to the naturally embedded $so(4)$ algebra as follows

$$31 \supset 1_{nc} \oplus 6_c \oplus 6_{nc} \oplus 9_c \oplus 9_{nc},$$

where c and nc denote the compact and noncompact operators respectively. The 6_c and 9_{nc} parts generate the $SL(4,R)$ subgroup of the $SL(4,C)$ group. The maximal compact subgroup $SO(4)$ of the $SL(4,R)$ group is realized in this embedding through its (single valued) vector representation $(\frac{1}{2},\frac{1}{2})$. In order to embed $\overline{SL}(4,R)$ into $SL(4,C)$, one would now have to embed its maximal compact subgroup $SO(4)$ in the maximal compact subgroup $SL(4,C)$, namely $SU(4)$. However, that is impossible $(\frac{1}{2},\frac{1}{2}) \not\to (\frac{1}{2},0)\oplus(0,\frac{1}{2})$. The alternative could have been to embed $\overline{SO}(4)$ in a hypothetical double covering of $SU(4)$ - except that $SU(4)$ is simply connected and thus is its own universal covering. We, therefore, conclude that in the *natural* embedding one can embed $SL(4,R)$ in $SL(4,C)$ *but not the* $\overline{SL}(4,R)$ *covering group.*

Dirac $\{(\frac{1}{2},0)\oplus(0,\frac{1}{2})\}$ embedding

In this case the $so(4)\to su(4)$ embedding is realized through a direct sum of 2×2 complex matrices, i.e.

$$so(4) \to \begin{pmatrix} su(2) & 0 \\ 0 & su(2) \end{pmatrix} \in su(4).$$

The $SL(4,C)$ generators now split with respect to the $su(2)\oplus su(2)$ algebra as follows

$$31 \supset 1_c \oplus 1_{nc} \oplus 1_{nc} \oplus 4_c \oplus 4_c \oplus 4_{nc} \oplus 4_{nc} \oplus 6_c \oplus 6_{nc}.$$

It is obvious from this decomposition that in the $sl(4,C)$ algebra *there exist no 9-component noncompact irreducible tensor-operator* with respect to the chosen $su(2)\oplus su(2) \sim so(4)$ subalgebra, which would, together with the 6_c operators, form an $sl(4,R)$ algebra. Thus, we conclude that this type of $so(4)$ embedding into $su(4)\subset sl(4,C)$ does not extend to an embedding of either $SL(4,R)$ of $\overline{SL}(4,R)$ into $SL(4,C)$:

To sum up, we have demonstrated that **there exist no finite-dimensional faithful representations of** $\overline{SL}(4,R)$.

4 The deunitarizing automorphism.

The unitarity properties, that ensure correct physical characteristics of the affine fields, can be achieved by combining the unitary (irreducible) representations and the so called "deunitarizing" automorphism of the $\overline{SL}(n,R)$ group.

The commutation relations of the $\overline{SL}(n,R)$ generators Q_{ab}, $a,b=0,1,\dots,n-1$ are

$$[Q_{ab},Q_{cd}] = i(\eta_{bc}Q_{ad} - \eta_{ad}Q_{cb}),$$

taking $\eta_{ab} = diag(+1, -1, \ldots, -1)$. The important subalgebras are as follows.

(i) $so(1, n-1)$: The $M_{ab} = Q_{[ab]}$ operators generate the Lorentz-like subgroup $\overline{SO}(1, n-1)$ with $J_{ij} = M_{ij}$ (angular momentum) and $K_i = M_{0i}$ (the boosts) $i, j = 1, 2, \ldots, n-1$.

(ii) $so(n)$: The J_{ij} and $N_i = Q_{\{0i\}}$ operators generate the maximal compact subgroup $\overline{SO}(n)$.

(iii) $sl(n-1)$: The J_{ij} and $T_{ij} = Q_{\{ij\}}$ operators generate the subgroup $\overline{SL}(n-1, R)$ - the "little" group of the massive particle states.

The $\overline{SL}(n, R)$ commutation relations are invariant under the "deunitarizing" automorphism,

$$\begin{aligned} J'_{ij} = J_{ij} \ , \quad K'_i = iN_i \ , \quad N'_i = iK_i \ , \\ T'_{ij} = T_{ij} \ , \quad T'_{00} = T_{00} \ (= Q_{00}) \ , \end{aligned}$$

so that $(J_{ij},\ iK_i)$ generate the new compact $\overline{SO}(n)'$ and $(J_{ij},\ iN_i)$ generate $\overline{SO}(1, n-1)'$.

For the massive (spinorial) particle states we use the basis vectors of the unitary irreducible representations of $\overline{SL}(n, R)'$, so that the compact subgroup finite multiplets correspond to $\overline{SO}(n)'$: $(J_{ij},\ iK_i)$ while $\overline{SO}(1, n-1)'$: $(J_{ij},\ iN_i)$ is represented by unitary infinite-dimensional representations. We now perform the inverse transformation and return to the unprimed $\overline{SL}(n, R)$ for our physical identification: $\overline{SL}(n, R)$ is represented non-unitarily, the compact $\overline{SO}(n)$ is represented by non-unitary infinite representations while the Lorentz group is represented by non-unitary finite representations. These finite-dimensional non-unitary Lorentz group representations are necessary in order to ensure a correct particle interpretation (i.g. boosted proton remains proton). Note that $\overline{SL}(n-1, R)$, the stability subgroup of $\overline{SA}(n, R)$, is represented unitarily.

5 World spinor field transformations.

The world spinor fields transform w.r.t. $\overline{Diff}(n, R)$ as follows

$(D(a, \bar{f})\Psi_M)(x) = (D_{\overline{Diff}_0(n,R)})^N_M(\bar{f})\Psi_N(f^{-1}(x-a))$,

$(a, \bar{f}) \in T_n \wedge \overline{Diff}_0(n, R)$,

where $\overline{Diff}_0(n, R)$ is the homogeneous part of $\overline{Diff}(n, R)$, and $D_{\overline{Diff}_0(n,R)} \supset \sum^{\oplus} D_{\overline{SL}(n,R)}$ is the corresponding representation in the space of world

spinor field components. As a matter of fact, we consider here those representations of $\overline{Diff}_0(n,R)$ that are nonlinearly realized over the maximal linear subgroup $\overline{SL}(n,R)$ (here given in terms of infinite matrices).

The affine "particle" states transform according to the following representation

$$D(a,\bar{s}) \rightarrow e^{i(sp)\cdot a} D_{\overline{SL}(n,R)}(L^{-1}(sp)\bar{s}L(p)), \quad (a,\bar{s}) \in T_n \wedge \overline{SL}(n,R),$$

where $L \in \overline{SL}(n,R)/\overline{SL}(n-1,R)$, and p is the n-momentum. The unitarity properties of various representations in these expressions is as described in the previous section.

Provided the relevant $\overline{SL}(n,R)$ representations are known, one can first define the corresponding general/special Affine spinor fields in the tangent to R^n, and than make use of the infinite-component pseudo-frame fields $E^A_M(x)$, "alephzeroads", that generalize the tetrad fields of R^4. Let us define a pseudo-frame $E^A_M(x)$ s.t.

$$\Psi_M(x) = E^A_M(x)\Psi_A(x),$$

where $\Psi_M(x)$ and $\Psi_A(x)$ are the world (holonomic), and general/special Affine spinor fields respectively. The $E^A_M(x)$ (and their inverses $E^M_A(x)$) are thus infinite matrices related to the quotient $\overline{Diff}_0(n,R)/\overline{SL}(n,R)$. Their infinitesimal transformations are

$$\delta E^A_M(x) = i\epsilon^a_b(x)\{Q^b_a\}^A_B E^B_M(x) + \partial_\mu \xi^\nu e^a_\nu e^\mu_b \{Q^a_b\}^A_B E^B_M(x),$$

where ϵ^a_b and ξ^μ are group parameters of $\overline{SL}(n,R)$ and $\overline{Diff}(n,R)/\overline{Diff}_0(n,R)$ respectively, while e^a_ν are the standard n-bine fields.

The infinitesimal transformations of the world spinor fields themselves are given as follows:

$$\delta\Psi^M(x) = i\{\epsilon^a_b(x)E^M_A(x)(Q^b_a)^A_B E^B_N(x) + \xi^\mu[\delta^M_N\partial_\mu + E^M_B(x)\partial_\mu E^B_N(x)]\}\Psi^N(x).$$

The $(Q^b_a)^M_N = E^M_A(x)(Q^b_a)^A_B E^B_N(x)$ is the holonomic form of the $\overline{SL}(n,R)$ generators given in terms of the corresponding anholonomic ones in the space of spinor fields $\Psi_M(x)$ and $\Psi_A(x)$ respectively.

The above outlined construction allows one to define a fully $\overline{Diff}(n,R)$ covariant Dirac-like wave equation for the corresponding world spinor fields provided a Dirac-like wave equation for the $\overline{SL}(n,R)$ group is known. In other words, one can lift an $\overline{SL}(n,R)$ covariant equation of the form

$$(ie^\mu_a(X^a)^B_A\partial_\mu - M)\Psi_B(x) = 0,$$

to a $\overline{Diff}(n,R)$ covariant equation

$$(ie^\mu_a E^N_B (X^a)^B_A E^A_M \partial_\mu - M)\Psi_N(x) = 0,$$

provided a spinorial $\overline{SL}(n,R)$ representation for the Ψ field is given, with the corresponding representation Hilbert space invariant w.r.t. X^a action. **Thus, the crucial step towards a Dirac-like world spinor equation is a construction of the corresponding $\overline{SL}(n,R)$ wave equation**.

6 $\overline{SL}(4,R)$ vector operator X.

For the construction of a Dirac-type equation, which is to be invariant under (special) affine transformations, we have two possible approaches to derive the matrix elements of the generalized Dirac matrices X_a.

We can consider the defining commutation relations of a $\overline{SL}(4,R)$ vector operator X_a,

$$[M_{ab}, X_c] = i\eta_{bc}X_a - i\eta_{ac}X_b$$
$$[T_{ab}, X_c] = i\eta_{bc}X_a + i\eta_{ac}X_b$$

One can obtain the matrix elements of the generalized Dirac matrices X_a by solving these relations for X_a in the Hilbert space of a suitable representation of $\overline{SL}(4,R)$. Alternatively, one can embed $\overline{SL}(4,R)$ into $\overline{SL}(5,R)$. Let the generators of $\overline{SL}(5,R)$ be ${R_A}^B$, $A,B = 0,...,4$. Now, there are two natural $\overline{SL}(4,R)$ four-vectors X_a, and Y_a defined by

$$X_a := R_{a4}, \quad Y_a := L_{4a}, \qquad a = 0,1,2,3.$$

The operator X_a (Y_a) obtained in this way fulfills the required $SL(4,R)$ four-vector commutation relations by construction. It is interesting to point out that the operator $G_a = \frac{1}{2}(X_a - Y_a)$ satisfies

$$[G_a, G_b] = -iM_{ab},$$

thereby generalizing a property of Dirac's γ-matrices. Since X_a, M_{ab} and T_{ab} form a closed algebra, the action of X_a on the $\overline{SL}(4,R)$ states does not lead out of the $\overline{SL}(5,R)$ representation Hilbert space.

In order to obtain an impression about the general structure of the matrix X_a, let us consider the following embedding of three finite-dimensional

tensorial $SL(4,R)$ irreducible representations into the corresponding one of $SL(5,R)$,

$$
\begin{array}{ccc}
SL(5,R) & \supset & SL(4,R) \\
\underbrace{\overset{15}{\square\square}}_{\varphi_{AB}} & \supset & \underbrace{\overset{10}{\square\square}}_{\varphi_{ab}} \oplus \underbrace{\overset{4}{\boxtimes\square}}_{\varphi_a} \oplus \underbrace{\overset{1}{\boxtimes\boxtimes}}_{\varphi} ,
\end{array}
$$

where "box" is the Young tableau for an irreducible vector representation of $SL(n,R)$, $n = 4,5$. The effect of the action of the $SL(4,R)$ vector X_a on the fields φ, φ_a and φ_{ab} is

$$
\overset{X_a}{\square} \otimes \overset{\varphi}{\boxtimes\boxtimes} \mapsto \overset{\varphi_a}{\square}, \qquad \overset{X_a}{\square} \otimes \overset{\varphi_a}{\boxtimes\square} \mapsto \overset{\varphi_{ab}}{\square\square}, \qquad \overset{X_a}{\square} \otimes \overset{\varphi_{ab}}{\square\square} \mapsto \; 0 \, .
$$

Other possible Young tableaux do not appear due to the closure of the Hilbert space. Gathering these fields in a vector $\varphi_M = (\varphi, \varphi_a, \varphi_{ab})^{\mathrm{T}}$, we can read off the structure of X_a,

$$
X_a = \left[\begin{array}{c|c|c}
0 & & \\
\hline
\begin{matrix} \mathrm{x} \\ \mathrm{x} \\ \mathrm{x} \\ \mathrm{x} \end{matrix} & \mathbf{0}_4 & \\
\hline
 & \begin{matrix} \mathrm{xxxx} \\ \mathrm{xxxx} \\ \mathrm{xxxx} \\ \mathrm{xxxx} \\ \mathrm{xxxx} \\ \mathrm{xxxx} \\ \mathrm{xxxx} \\ \mathrm{xxxx} \\ \mathrm{xxxx} \\ \mathrm{xxxx} \end{matrix} & \mathbf{0}_{10}
\end{array} \right] .
$$

It is interesting to observe that X_a has zero matrices on the block-diagonal which implies that the mass operator κ in an affine invariant equation must vanish.

This can be proven for a general finite representation of $SL(4,R)$. Let us consider the action of a vector operator on an arbitrary irreducible representation $D(g)$ of $SL(4,R)$ labeled by $[\lambda_1, \lambda_2, \lambda_3]$, λ_i being the number of boxes in the i-th raw,

$$
\begin{aligned}
[\lambda_1, \lambda_2, \lambda_3] \otimes [1,0,0] = \quad & [\lambda_1 + 1, \lambda_2, \lambda_3] \oplus [\lambda_1, \lambda_2 + 1, \lambda_3] \; \oplus \\
& [\lambda_1, \lambda_2, \lambda_3 + 1] \oplus [\lambda_1 - 1, \lambda_2 - 1, \lambda_3 - 1] \, .
\end{aligned}
$$

None of the resulting representations agrees with the representation $D(g)$ nor with the *contragradient* representation $D^{\mathrm{T}}(g^{-1})$ given by

$$
[\lambda_1, \lambda_2, \lambda_3]^{\mathrm{c}} = [\lambda_1, \lambda_1 - \lambda_3, \lambda_1 - \lambda_2] \, .
$$

For a general (reducible) representation this implies vanishing matrices on the block-diagonal of X_a by similar argumentation as that that led to the structure of X_a. Let the representation space be spanned by $\Phi = (\varphi_1, \varphi_2, ...)^{\mathrm{T}}$ with φ_i irreducible. Now we consider the Dirac-type equation in the rest frame $p_\mu = (E, 0, 0, 0)$ restricted to the subspaces spanned by φ_i ($i = 1, 2, \ldots$),

$$E < \varphi_i, X^0 \varphi_j > = < \varphi_i, M \varphi_i > = \; m_i \delta_{ij},$$

where we assumed the operator M to be diagonal. So the mass m_i and therewith M must vanish since $< \varphi_i, X^0 \varphi_i >= 0$. Therefore, **in an affine invariant Dirac-type wave equation the mass generation can only be dynamical**, i.e. a result of an interaction. This agrees with the fact that the Casimir operator of the special affine group $\overline{SA}(4, R)$ vanishes leaving the masses unconstrained; thus we expect that our statement also holds for infinite representations of $\overline{SL}(4, R)$ as well.

7 $D = 3$ Vector operator

Let us consider now the $\overline{SL}(3, R)$ spinorial representations, that are necessarily infinite-dimensional. There is a unique multiplicity-free ("ladder") unitary irreducible representation of the $\overline{SL}(3, R)$ group, $D^{(ladd)}_{\overline{SL}(3,R)}(\frac{1}{2})$, that in the reduction w.r.t. its maximal compact subgroup $\overline{SO}(3)$ yields,

$$D^{(ladd)}_{\overline{SL}(3,R)}(\frac{1}{2}) \supset D^{(\frac{1}{2})}_{\overline{SO}(3)} \oplus D^{(\frac{5}{2})}_{\overline{SO}(3)} \oplus D^{(\frac{9}{2})}_{\overline{SO}(3)} \oplus \ldots$$

i.e. it has the following J content: $\{J\} = \{\frac{1}{2}, \; \frac{5}{2}, \; \frac{9}{2}, \; \ldots\}$.

Owing to the fact that the $\overline{SO}(3)$ and/or $\overline{SL}(3, R)$ vector operators can have nontrivial matrix elements only between the $\overline{SO}(3)$ states such that $\Delta J = 0, \pm 1$, it is obvious (on account of the Wigner-Eckart theorem) that all X-operator matrix elements *vanish* for the Hilbert space of the $D^{(ladd)}_{\overline{SL}(3,R)}(\frac{1}{2})$ representation (where $\Delta J = \pm 2$). The same holds for the two classes of tensorial ladder unitary irreducible representations $D^{(ladd)}_{\overline{SL}(3,R)}(0; \sigma_2)$ and $D^{(ladd)}_{\overline{SL}(3,R)}(1; \sigma_2)$, $\sigma_2 \in R$, with the J content $\{J\} = \{0, 2, 4, \ldots\}$ and $\{J\} = \{1, 3, 5, \ldots\}$.

Let us consider now the case of $\overline{SL}(3, R)$ unitary irreducible representations with nontrivial multiplicity w.r.t. the maximal compact subgroup $\overline{SO}(3)$. An efficient way to construct these representations explicitly is to

set up a Hilbert space of square-integrable functions $H = L^2([\overline{SO}(3) \otimes \overline{SO}(3)]^d, \kappa)$, over the diagonal subgroup of the two copies of the $\overline{SO}(3)$ subgroup, with the group action to the right defining the group/representation itself while the group action to the left accounts for the multiplicity. Here, κ stands for the scalar product kernel, that has to be more singular than the Dirac delta function in order to account for all types of $\overline{SL}(3,R)$ unitary irreducible representations. We consider the canonical (spherical) basis of this space $\sqrt{2J+1}D^J_{KM}(\alpha,\beta,\gamma)$, where J and M are the representation labels defined by the subgroup chain $\overline{SO}(3) \supset \overline{SO}(2)$, while K is the label of the extra copy $\overline{SO}(2)_L \subset \overline{SO}(3)_L$ that describes nontrivial multiplicity. Here, $-J \leq K, M \leq +J$, and for each allowed K one has $J \geq K$, i.e. $J = K, K+1, K+2, \ldots$.

A generic 3-vector operator ($J = 1$) is given now in the spherical basis ($\alpha = 0, \pm 1$) by

$$X_\alpha = \mathcal{X}_{(0)} D^{(1)}_{0\alpha}(k) + \mathcal{X}_{(\pm 1)}[D^{(1)}_{+1\alpha}(k) + D^{(1)}_{-1\alpha}(k)], \quad k \in \overline{SO}(3).$$

The corresponding matrix elements between the states of two unitary irreducible $\overline{SL}(3,R)$ representations that are characterized by the labels σ and δ are given as follows:

$$\left\langle \begin{array}{c} (\sigma'\ \delta') \\ J' \\ K'\ M' \end{array} \middle| X_\alpha \middle| \begin{array}{c} (\sigma\ \delta) \\ J \\ K\ M \end{array} \right\rangle = (-)^{J'-K'} (-)^{J'-M'} \sqrt{(2J'+1)(2J+1)}$$
$$\times \begin{pmatrix} J' & 1 & J \\ -M' & \alpha & M \end{pmatrix} \left\{ \mathcal{X}_{(0)}{}^{(\sigma'\delta'\sigma\delta)}_{J'J} \begin{pmatrix} J' & 1 & J \\ -K' & 0 & K \end{pmatrix} \right.$$
$$\left. + \mathcal{X}_{(\pm 1)}{}^{(\sigma'\delta'\sigma\delta)}_{J'J} \left[\begin{pmatrix} J' & 1 & J \\ -K' & 1 & K \end{pmatrix} + \begin{pmatrix} J' & 1 & J \\ -K' & -1 & K \end{pmatrix} \right] \right\}.$$

Therefore, the action of a generic $\overline{SL}(3,R)$ vector operator on the Hilbert space of some nontrivial-multiplicity unitary irreducible representation produces the $\Delta J = 0, \pm 1$, as well as the $\Delta K = 0, \pm 1$ transitions. Owing to the fact that the states of a unitary irreducible $\overline{SL}(3,R)$ representation are characterized by the $\Delta K = 0, \pm 2$ condition, it is clear that the $\Delta K = \pm 1$ transitions due to 3-vector X, can take place only between the states of mutually inequivalent $\overline{SL}(3,R)$ representations whose multiplicity is characterized by the K values of opposite evenness. In analogy to the finite-dimensional (tensorial) representation case, the repeated applications of a vector operator on a given unitary irreducible (spinorial and/or tensorial) $\overline{SL}(3,R)$ representation would yield, a priori, an infinite set of

irreducible representations. Due to an increased mathematical complexity of the infinite-dimensional representations, some additional algebraic constraints imposed on the vector operator X would be even more desirable than in the finite-dimensional case. The most natural option is to embed the $\overline{SL}(3,R)$ 3-vector X together with the $\overline{SL}(3,R)$ algebra itself into the (simple) Lie algebra of the $\overline{SL}(4,R)$ group. Any spinorial (and/or tensorial) $\overline{SL}(4,R)$ unitary irreducible representation provides now a Hilbert space that can be decomposed w.r.t. $\overline{SL}(3,R)$ subgroup representations, and most importantly this space is, by construction, invariant under the action of the vector operator X. Moreover, an explicit construction of the starting $\overline{SL}(4,R)$ representation generators would yield an explicit form of the X operator.

8 Embedding into $\overline{SL}(4,R)$

The $\overline{SL}(4,R)$ group is a 15-parameter non-compact Lie group whose defining (spinorial) representation is given in terms of infinite matrices. All spinorial (unitary and nonunitary) representations of $\overline{SL}(4,R)$ are necessarily infinite-dimensional; the finite-dimensional tensorial representations are nonunitary, while the unitary tensorial representations are infinite-dimensional. The $\overline{SL}(4,R)$ commutation relations in the Minkowski space are given by,

$$[Q_{ab}, Q_{cd}] = i\eta_{bc}Q_{ad} - i\eta_{ad}Q_{cb}.$$

where, $a,b,c,d = 0,1,2,3,$ and $\eta_{ab} = diag(+1,-1,-1,-1)$, while in the Euclidean space they read,

$$[Q_{ab}, Q_{cd}] = i\delta_{bc}Q_{ad} - i\delta_{ad}Q_{cb}.$$

where, $a,b,c,d = 1,2,3,4,$ and $\delta_{ab} = diag(+1,+1,+1,+1)$

The relevant subgroup chain reads:

$$\begin{array}{ccc} \overline{SL}(4,R) & \supset & \overline{SL}(3,R) \\ \cup & & \cup \\ \overline{SO}(4), \overline{SO}(1,3) & \supset & \overline{SO}(3), \overline{SO}(1,2). \end{array}$$

We denote by R_{mn}, $(m,n = 1,2,3,4)$ the 6 compact generators of the maximal compact subgroup $\overline{SO}(4)$ of the $\overline{SL}(4,R)$ group, and the remaining 9 noncompact generators (of the $\overline{SL}(4,R)/\overline{SO}(4)$ coset) by Z_{mn}.

In the $\overline{SO}(4) \simeq SU(2) \otimes SU(2)$ spherical basis, the compact operators are $J^{(1)}_i = \frac{1}{2}(\epsilon_{ijk}R_{jk} + R_{i4})$ and $J^{(2)}_i = \frac{1}{2}(\epsilon_{ijk}R_{jk} - R_{i4})$, while the noncompact generators we denote by $Z_{\alpha\beta}$, $(\alpha,\beta = 0,\pm 1)$, and they transform as

a $(1,1)$-tensor operator w.r.t. $SU(2)\otimes SU(2)$ group. The minimal set of commutation relations in the spherical basis reads:

$$
\begin{aligned}
&[J_0^{(p)}, J_\pm^{(q)}] = \pm\delta_{pq}J_\pm^{(p)}, \quad [J_+^{(p)}, J_-^{(q)}] = 2\delta_{pq}J_0^{(p)}, \quad (p,q=1,2),\\
&[J_0^{(1)}, Z_{\alpha\beta}] = \alpha Z_{\alpha\beta}, \quad [J_\pm^{(1)}, Z_{\alpha\beta}] = \sqrt{2-\alpha(\alpha\pm 1)}Z_{\alpha\pm1\ \beta}\\
&[J_0^{(2)}, Z_{\alpha\beta}] = \beta Z_{\alpha\beta}, \quad [J_\pm^{(2)}, Z_{\alpha\beta}] = \sqrt{2-\beta(\beta\pm 1)}Z_{\alpha\beta\pm1}\\
&[Z_{+1\ +1}, Z_{-1\ -1}] = -(J^{(1)}+J^{(2)}).
\end{aligned}
$$

The $\overline{SO}(3)$ generators are $J_i = \epsilon_{ijk}J_{jk},\ J_{ij}\equiv R_{ij},\ (i,j,k=1,2,3)$, while the traceless $T_{ij} = Z_{ij}\ (i,j=1,2,3)$ define the coset $\overline{SL}(3,R)/\overline{SO}(3)$. In the $\overline{SO}(3)$ spherical basis the compact operators are $J_0,\ J_{\pm1}$, while the non-compact ones $T_\rho,\ (\rho=0,\pm1,\pm2)$ transform w.r.t. $\overline{SO}(3)$ as a quadrupole operator. The corresponding minimal set of commutation relations reads:

$$
\begin{aligned}
&[J_0, J_\pm] = \pm J_\pm, \quad [J_+, J_-] = 2J_0\\
&[T_{+2}, T_{-2}] = -4J_0.
\end{aligned}
$$

There are three (independent) $\overline{SO}(3)$ vectors in the algebra of the $\overline{SL}(4,R)$ group. They are: the $\overline{SO}(3)$ generators themselves, $N_i\equiv R_{i4} = Q_{i0}+Q_{0i}$, and $K_i\equiv Z_{i4} = Q_{i0}-Q_{0i}$. From the latter two, one can form the following linear combinations,

$$
A_i = \frac{1}{2}(N_i+K_i) = Q_{i0}, \quad B_i = \frac{1}{2}(N_i-K_i) = Q_{0i}.
$$

The commutation relations between N, K, A, and B and the $\overline{SL}(3,R)$ generators read:

$$
\begin{aligned}
&[J_i, N_j] = i\epsilon_{ijk}N_k, \quad [T_{ij}, N_k] = i(\delta_{ik}K_j+\delta_{jk}K_i),\\
&[J_i, K_j] = i\epsilon_{ijk}K_k, \quad [T_{ij}, K_k] = i(\delta_{ik}N_j+\delta_{jk}N_i),\\
&[J_i, A_j] = i\epsilon_{ijk}A_k, \quad [T_{ij}, A_k] = i(\delta_{ik}A_j+\delta_{jk}A_i),\\
&[J_i, B_j] = i\epsilon_{ijk}B_k, \quad [T_{ij}, B_k] = -i(\delta_{ik}B_j+\delta_{jk}B_i).
\end{aligned}
$$

It is clear from these expressions that only A_i and B_i are $\overline{SL}(3,R)$ vectors as well. More precisely, A transforms w.r.t. $\overline{SL}(3,R)$ as the 3-dimensional representation $[1,0]$, while B transforms as its contragradient 3-dimensional representation $[1,1]$.

To summarize, either of the choices

$$
X_i \sim A_i, \quad X_i \sim B_i
$$

insures that a Dirac-like wave equation $(iX\partial - m)\Psi(x) = 0$ for a (infinite-component) spinor field is fully $\overline{SL}(3,R)$ covariant. The choices

$$X_i \sim J_i, \quad X_i \sim N_i, \quad X_i \sim K_i,$$

would yield wave equations that are Lorentz covariant only; though the complete $\overline{SL}(3,R)$ acts invariantly in the space of $\Psi(x)$ components. It goes without saying that the correct unitarity properties can be accounted for by making use of the deunitarizing automorphism, as discussed above.

Due to complexity of the generic unitary irreducible representations of the $\overline{SL}(4,R)$ group, we confine here to the multiplicity-free case only. In this case, there are just two, mutually contragradient, representations that contain spin $J = \frac{1}{2}$ representation of the $\overline{SO(3)}$ subgroup, and belong to the set of the so called Descrete Series i.e.

$$D^{disc}_{\overline{SL}(4,R)}(\frac{1}{2},0) \supset D^{(\frac{1}{2},0)}_{\overline{SO}(4)} \supset D^{\frac{1}{2}}_{\overline{SO}(3)} \quad D^{disc}_{\overline{SL}(4,R)}(0,\frac{1}{2}) \supset D^{(0,\frac{1}{2})}_{\overline{SO}(4)} \supset D^{\frac{1}{2}}_{\overline{SO}(3)}.$$

The full reduction of these representations to the representations of the $\overline{SL}(3,R)$ subgroup reads:

$$\begin{aligned} D^{disc}_{\overline{SL}(4,R)}(j_0,0) &\rightarrow \Sigma^{\oplus\infty}_{j=1} D^{disc}_{\overline{SL}(3,R)}(j_0;\sigma_2,\delta_1,j) \\ D^{disc}_{\overline{SL}(4,R)}(0,j_0) &\rightarrow \Sigma^{\oplus\infty}_{j=1} D^{disc}_{\overline{SL}(3,R)}(j_0;\sigma_2,\delta_1,j) \end{aligned}$$

Finally, by making use of the known expressions of the $\overline{SL}(4,R)$ generators matrix elements for these spinorial representations, we can write down an $\overline{SL}(3,R)$ covariant wave equation in the form

$$\begin{aligned} &(iX^\mu \partial_\mu - M)\Psi(x) = 0, \\ &\Psi \ \sim \ D^{disc}_{\overline{SL}(4,R)}(\frac{1}{2},0) \oplus D^{disc}_{\overline{SL}(4,R)}(0,\frac{1}{2}), \\ &X^\mu \in \{Q^{\mu 0},\ Q^{0\mu}\}. \end{aligned}$$

References

[1] Dj. Šijački, *J. Math. Phys.* **16** (1975) 298.

[2] Y. Ne'eman and Dj. Šijački, *Ann. Phys. (N.Y.)* **120** (1979) 292.

[3] J. Mickelsson, *Commun. Math. Phys.* **88** (1983) 551.

[4] Dj. Šijački and Y. Ne'eman, *J. Math. Phys.* **26** (1985) 2475.

[5] A. Cant and Y. Ne'eman, *J. Math. Phys.* **26** (1985) 3180.

[6] Y. Ne'eman and Dj. Šijački, *Phys. Lett.* **B 157** (1985) 267.

[7] Y. Ne'eman and Dj. Šijački, *Phys. Lett.* **B 157** (1985) 275.

[8] Y. Ne'eman and Dj. Šijački, *Int. J. Mod. Phys.* **A 2** (1987) 1655.

[9] Y. Ne'eman and Dj. Šijački, *Found. Phys* **27** (1997) 1105.

[10] Dj. Šijački, *Acta Phys. Polonica* **B 29**M (1998) 1089.